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Letters to the Editor

Chebyshev Approximation with Respect to a Vanishing Weight Function

Communicated by John R. Rice

Let $[\alpha, \beta]$ be a closed interval and $\| \ \|$ the Chebyshev norm on $[\alpha, \beta]$. Let s be a nonnegative continuous function which does not vanish identically. Let (F, P) be a family of continuous functions on $[\alpha, \beta]$ such that best unweighted Chebyshev approximations are characterized by alternation. The approximation problem is: given f continuous on $[\alpha, \beta]$, to find a parameter $A^* \in P$ minimizing $e(A) = \| s(f - F(A, \cdot))\|$ over $A \in P$. Such a parameter A^* is called best.

This problem has been completely solved when s does not vanish [1], but apparently has never been examined for the case in which s does vanish. It turns out to be no more difficult to study the case of a vanishing generalized weight function [1]. Let R be the real line.

DEFINITION. A continuous mapping w of $[\alpha, \beta] \times R \times R$ into the extended real line is called a *weak ordering function* if for given $x \in [\alpha, \beta]$, either

(i) $w(x, \cdot, \cdot) = 0$, or

(ii) for fixed a, $w(x, a, b)$ is a monotonic function of b (strictly monotonic when it is finite) and

$$\operatorname{sgn}(w(x, a, b)) = \operatorname{sgn}(a - b).$$

Further (i) does not happen for all $x \in [\alpha, \beta]$.

An examination of [1] shows that the theory developed there holds for approximation with respect to a weak ordering function, and the Remez algorithm can be used to determine the best approximation (provided we do not use as initial x_i points where w vanishes identically).

For the approximation problem raised in the first paragraph, we obtain the following:

THEOREM. *Let s be a nonnegative continuous function. Let F have degree n*

at A. A necessary and sufficient condition that A be best to f with respect to weight function s is that $s(f - F(A, \cdot))$ *alternate n times.*

Further, best approximations are unique [1, middle of page 226].

Reference

1. C. B. Dunham, Chebyshev approximation with respect to a weight function, *J. Approximation Theory* **2** (1969), 223–232.

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